# Convergence of random walks on compact Lie groups to equilibrium

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## 1. Introduction

The goal of this note is to explain how to prove distributional convergence of random walks on compact Lie groups to the Haar measure. We refer to [1] which explains the underlying theory very nicely but goes in a different and in particular much more quantitative direction than this note. The main message here is that large parts of probability theory for real-valued random processes rely on Fourier transforms to understand distributional asymptotics (mixing behaviors being an example). If we are now looking at random processes on a Lie group, the natural thing to do is to look at irreducible representations and these will form the analogous objects to the Fourier modes on the real line.

## 2. Representation theory review

Suppose that $G$ is a compact Lie group and $\mu$ is the Haar measure on $G$. We label all the irreducible representations of $G$ by $\hat{G}$, that is

$$\pi_\lambda : G \to U(V_\lambda)$$

is an irreducible representations from $G$ on a Hilbert space $V_\lambda$ of dimension $D_\lambda$ for any $\lambda \in \hat{G}$. The key result is then the following. Below $\mathrm{End}(V_\lambda)$ denotes the space of linear maps from $V_\lambda$ to itself and we endow this vector space with the inner product $\langle \varphi, \psi \rangle_{\mathrm{End}(V_\lambda)} = D_\lambda \operatorname{tr}\left(\varphi^\dagger \psi\right)$ for $\varphi, \psi \in \mathrm{End}(V_\lambda)$.

**Theorem 2.1 (Peter-Weyl).** The map $\mathcal{F}$ defined below is a linear isometry and we also have the explicit formula for $\mathcal{F}^{-1}$ given below.

$$\mathcal{F} : L^2(G, \mu) \to \ell^2\left( (\mathrm{End}(V_\lambda))_{\lambda \in \hat{G}} \right),$$

$$\mathcal{F} f = \left( \int \pi_\lambda(g) f(g) \mu(dg) \right)_\lambda,$$

$$\mathcal{F}^{-1}(a_\lambda)_\lambda = \left( g \mapsto \sum_{\lambda \in \hat{G}} D_\lambda \operatorname{tr}\left( a_\lambda \pi_\lambda^\dagger(g) \right) \right).$$

We will write $\pi_{\lambda_0}$ for the trivial representation with character $\operatorname{tr}\left(\pi_{\lambda_0}(g)\right) = 1$ for all $g \in G$ and note character orthogonality, namely

$$\int \operatorname{tr}(\pi_\lambda(g)) \operatorname{tr}\left(\pi_{\lambda'}^\dagger(g)\right) \mu(dg) = \delta_{\lambda\lambda'}$$

and so in particular

$$\int \operatorname{tr}(\pi_\lambda(g)) \mu(dg) = \delta_{\lambda\lambda_0}.$$

## 3. Random walks on groups

So let us now consider some measure $\nu$ on $G$ and let $(A_n)$ be i.i.d. copies of $\nu$, and let $X_0 = e$ and $X_n = A_n \cdots A_1$ for $n \geq 1$ where $e$ denotes the identity element of $G$. Consider $f \in L^2(G, \mu)$ which we write using the Peter-Weyl theorem as

$$f(g) = \sum_{\lambda \in \hat{G}} D_\lambda \operatorname{tr}\Big((\mathcal{F} f)_\lambda \pi_\lambda^\dagger(g)\Big).$$

Ignoring any issues about interchanging limits in the following two paragraphs, we therefore obtain that we have

$$\begin{aligned} \mathbb{E}(f(X_n)) &= \sum_{\lambda \in \hat{G}} D_\lambda \operatorname{tr}\Big((\mathcal{F} f)_\lambda \mathbb{E}\Big(\pi_\lambda^\dagger(A_n \cdots A_1)\Big)\Big) \\ &= \sum_{\lambda \in \hat{G}} D_\lambda \operatorname{tr}\Big((\mathcal{F} f)_\lambda \mathbb{E}\Big(\pi_\lambda^\dagger(A_1)\Big)^n\Big). \end{aligned}$$

It follows that if $\|\mathbb{E}(\pi_\lambda(A_1))\|_{\mathrm{op}} < 1$ for all $\lambda \neq \lambda_0$ then

$$\mathbb{E}(f(X_n)) \to \operatorname{tr}\Big((\mathcal{F} f)_{\lambda_0}\Big) = \int f(g)\mu(dg) \quad \text{as} \quad n \to \infty.$$

Making the above formal yields the following proposition.

**Proposition 3.1.** The following two assertions are equivalent: (i) We have $\| \int \pi_\lambda(g)\nu(dg)\|_{\mathrm{op}} < 1$ for all $\lambda \neq \lambda_0$, and (ii) the law of $X_n$ converges weakly to $\mu$ as $n \to \infty$.

*Proof.* Consider the set

$$\mathcal{S} = \left\{ \left( g \mapsto \sum_{\lambda \in F} D_\lambda \operatorname{tr}\Big(a_\lambda \pi_\lambda^\dagger(g)\Big) \right) : a_\lambda \in \operatorname{End}(V_\lambda) \, \forall \lambda \in F \text{ with } F \subseteq \hat{G} \text{ finite} \right\}.$$

Using the Peter-Weyl theorem it is not difficult to check that $\mathcal{S}$ satisfies the conditions of the Stone-Weierstrass theorem and is therefore dense in $(C(G), \| \cdot \|_\infty)$.

Let us first show that (i) implies (ii): By standard probabilistic results and the compactness of $G$, it is enough to show that any subsequential limit of the laws of the $X_n$ random variables equals $\mu$; let us therefore write $\mu'$ for such a subsequential limit. For elements $f \in \mathcal{S}$ the discussion just before the proposition is rigorous and it follows that

$$\int f(g)\mu'(dg) = \int f(g)\mu(dg) \quad \forall f \in \mathcal{S}.$$

The fact that $\mathcal{S}$ is dense in $(C(G), \| \cdot \|_\infty)$ implies that this extends to all $f \in C(G)$ and hence $\mu' = \mu$ by standard measure theory results as required.

To see that (ii) implies (i), suppose that $\| \int \pi_\lambda(g)\nu(dg)\|_{\mathrm{op}} = 1$ for some $\lambda \neq \lambda_0$ and set $f(g) = \operatorname{tr}\Big(\pi_\lambda^\dagger(g)\Big)$ so that $f$ is a continuous function on $G$. Then

$$\mathbb{E}(f(X_n)) = \operatorname{tr}\left(\left(\int \pi_\lambda^\dagger(g)\nu(dg)\right)^n\right) \not\to 0 = \int \operatorname{tr}\Big(\pi_\lambda^\dagger(g)\mu(dg)\Big) = \int f(g)\mu(dg)$$

as $n \to \infty$ which completes the proof. □

Recall that $\nu^{*k}$ denotes the $k$-fold convolution of the measure $\nu$ with respect to the group structure on $G$, so $\nu^{*k}$ is the law of $X_k$.

**Corollary 3.2.** If $\operatorname{supp}(\nu^{*k}) = G$ for some $k \geq 1$ then the law of $X_n$ converges weakly to $\mu$ as $n \to \infty$.

*Proof.* We have $\| \int \pi_\lambda(g) \nu^{*k}(dg) \|_{\text{op}} = \| \int \pi_\lambda(g) \nu(dg) \|_{\text{op}}^k$ so it is enough to show that the left hand side of this equality is $< 1$ for $\lambda \neq \lambda_0$. If it was $= 1$, since by assumption $\operatorname{supp}(\nu^{*k}) = 1$, this would imply by continuity of $\pi_\lambda$ that there is a $v \in V_\lambda \setminus \{0\}$ such that $\pi_\lambda(g)v = v$ for all $g \in G$ (note that the eigenvalue has to be 1 due to the $g = e$ case). Thus $\langle v \rangle$ is an invariant subspace of $V_\lambda$ and by irreducibility $D_\lambda = 1$. Since $\pi_\lambda$ is then the identity, in fact $\lambda = \lambda_0$. □

It is not hard to check that if $\operatorname{supp}(\nu)$ has non-empty interior and $G$ is connected, then the condition in the result is satisfied. To see this, one might want to use that the exponential map is surjective whenever $G$ is connected.

## References


[1] P.-L. Méliot, “The Cut-off Phenomenon for Brownian Motions on Compact Symmetric Spaces”, *Potential Anal.*, vol. 40, no. 4, pp. 427–509, 2013, doi: 10.1007/s11118-013-9356-7. Available: https://arxiv.org/abs/1210.0480